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Representations for Moore-Penrose Inverses in Hilbert Spaces

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Abstract—Let H_1, H_2 be two Hilbert spaces, and let $T\ H_1 \to H_2$ be a bounded linear operator with closed range We present some representations of the perturbation for the Moore-Penrose inverse in Hilbert spaces for the case that the perturbation does not change the range or the null space of the operator

Keywords—Perturbation, Moore-Penrose inverse, Range space, Null space

1. INTRODUCTION

Let H_1, H_2 be two Hilbert spaces over the same complex field, and let $L(H_1, H_2)$ denote the Banach space of all bounded linear operators $T\ H_1 \to H_2$ with the operator norm $\|T\| = \sup\{\|Tx\|\ \ \|x\| = 1\}$

For $T \in L(H_1, H_2)$ let $R(T)$ be the range of T and $N(T)$ be the null space of T According to [1,2], $T \in L(H_1, H_2)$ with $R(T)$ closed has the Moore-Penrose inverse T^+, namely, $T^+ \in L(H_2, H_1)$ is the unique solution of the following equations

$$TT^+T = T, \qquad T^+TT^+ = T^+, \qquad (TT^+)^* = TT^+, \qquad (T^+T)^* = T^+T, \tag{1}$$

in which T^* denotes the adjoint operator of T

Ding and Huang [3] presented the error estimate of the Moore-Penrose inverse for range or null space preserving perturbations of operators in Hilbert spaces But they did not give an explicit formula for the generalized inverse of the perturbed operator

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In this paper, we present some representations for the perturbation of Moore-Penrose inverses in Hilbert spaces for the case that the perturbation does not change the range or the null space of the operator Our expressions for the perturbed generalized inverse and error estimates extend earlier results in [3,4], in particular giving some substantial new insight and information in the case of matrices, that was not already given in [4]

2. MAIN RESULTS

Let $T \in L(H_1, H_2)$ with closed range and let $\tilde{T} = T + \delta T \in L(H_1, H_2)$ First, we consider this problem under what conditions $\tilde{T}^+ = (I + T^+\delta T)^{-1}T^+$ or $\tilde{T}^+ = T^+(I + \delta T T^+)^{-1}$?

LEMMA 1 *Suppose that* $R(\delta T) \subseteq R(T)$ *and* $\|T^+\delta T\| < 1$ *Then for* $A = (I + T^+\delta T)^{-1}T^+$,

$$\tilde{T}A\tilde{T} = \tilde{T}, \qquad A\tilde{T}A = A, \qquad \left(\tilde{T}A\right)^* = \tilde{T}A$$

PROOF Since $R(\delta T) \subseteq R(T)$, $\tilde{T} = T + \delta T = T(I + T^+\delta T)$ Since $\|T^+\delta T\| < 1$, A is well defined Since

$$\tilde{T}A = T\left(I + T^+\delta T\right)\left(I + T^+\delta T\right)^{-1}T^+ = TT^+,$$

$(\tilde{T}A)^* = (TT^+)^* = TT^+ = \tilde{T}A$, and

$$\tilde{T}A\tilde{T} = TT^+T\left(I + T^+\delta T\right) = T\left(I + T^+\delta T\right) = \tilde{T},$$

$$A\tilde{T}A = \left(I + T^+\delta T\right)^{-1}T^+TT^+ = \left(I + T^+\delta T\right)^{-1}T^+ = A$$

∎

Similarly, we have the following

LEMMA 2 *Suppose that* $N(T) \subseteq N(\delta T)$ *and* $\|\delta T T^+\| < 1$ *Then for* $A = T^+(I + \delta T T^+)^{-1}$,

$$\tilde{T}A\tilde{T} = \tilde{T}, \qquad A\tilde{T}A = A, \qquad \left(A\tilde{T}\right)^* = A\tilde{T}$$

REMARK 1 Using the same idea as in the proof of Lemmas 3 3 and 3 4 of [3], we see that in Lemma 1, $(I + \delta T T^+)^{-1}$ also exists and

$$\left(I + T^+\delta T\right)^{-1}T^+ = T^+\left(I + \delta T T^+\right)^{-1},$$

and in Lemma 2, $(I + T^+\delta T)^{-1}$ also exists and

$$T^+\left(I + \delta T T^+\right)^{-1} = \left(I + T^+\delta T\right)^{-1}T^+$$

THEOREM 1 *Suppose that* $R(\delta T) \subseteq R(T)$ *and* $N(T) \subseteq N(\delta T)$ *If* $\|T^+\delta T\| < 1$ *or* $\|\delta T T^+\| < 1$, *then* $\tilde{T}^+$ *is well defined and*

$$\tilde{T}^+ = \left(I + T^+\delta T\right)^{-1}T^+ = T^+\left(I + \delta T T^+\right)^{-1} \tag{2}$$

Moreover,

$$\frac{\left\|\tilde{T}^+ - T^+\right\|}{\|T^+\|} \le \frac{\|T^+\delta T\|}{1 - \|T^+\delta T\|} \tag{3}$$

or

$$\frac{\left\|\tilde{T}^+ - T^+\right\|}{\|T^+\|} \le \frac{\|\delta T T^+\|}{1 - \|\delta T T^+\|}, \tag{4}$$

depending on whether $\|T^+\delta T\| < 1$ *or* $\|\delta T T^+\| < 1$, *respectively*

PROOF Equation (2) follows by combining the above two lemmas, and (3) and (4) are from Lemma 3 1 of [3] ∎

COROLLARY 1 *(See [4]) Let $T \in C^{m\times n}$ and let $\tilde{T} = T + \delta T \in C^{m\times n}$ be such that $TT^+\delta T = \delta T$, $T^+T(\delta T)^* = (\delta T)^*$, and $\|T^+\|\ \|\delta T\| < 1$ Then the conclusion of Theorem 1 is true*

PROOF $TT^+\delta T = \delta T$ is equivalent to $R(\delta T) \subseteq R(T)$ and $T^+T(\delta T)^* = (\delta T)^*$ is equivalent to $N(T) \subseteq N(\delta T)$ ∎

REMARK 2 Therefore, Theorem 1 is a direct generalization of the main result of [4] to the case of infinite dimensional Hilbert spaces In fact, since our argument is only based on the Neumann Lemma (Lemma 3 1 of [3]) for more general Banach spaces in functional analysis, Theorem 1 and Corollary 1 are still valid for generalized inverses of bounded linear operators defined on general Banach spaces

REMARK 3

(i) Under the condition of Lemma 1,

$$A\tilde{T} = \left(I + T^+\delta T\right)^{-1} T^+T\left(I + T^+\delta T\right)$$

So a sufficient condition for $\tilde{T}^+ = (I + T^+\delta T)^{-1}T^+$ is that $I + T^+\delta T$ is unitary

(ii) Under the condition of Lemma 2,

$$\tilde{T}A = \left(I + \delta TT^+\right)TT^+\left(I + \delta TT^+\right)^{-1}$$

Thus, a sufficient condition for $\tilde{T}^+ = T^+(I + \delta TT^+)^{-1}$ is that $I + \delta TT^+$ is unitary

Now we present general expressions for $\tilde{T}^+$ when it is only assumed that $R(\delta T) \subseteq R(T)$ or $N(T) \subseteq N(\delta T)$

THEOREM 2 *Suppose that $R(\delta T) \subseteq R(T)$ and $\|T^+\delta T\| < 1$ Then $\tilde{T}^+$ is well defined, $\tilde{T}\tilde{T}^+ = TT^+$, and*

$$\begin{aligned}\tilde{T}^+ &= \left(I + T^+\delta T\right)^{-1}T^+ + \left(I + T^+\delta T\right)^{-1}\Big\{I + \left[\left(I + T^+\delta T\right)^{-1}T^+\delta T\left(I - T^+T\right)\right]^* \\ &\quad \left[\left(I + T^+\delta T\right)^{-1}T^+\delta T\left(I - T^+T\right)\right]\Big\}^{-1}\left(I - T^+T\right) \\ &\quad \left[\left(I + T^+\delta T\right)^{-1}T^+\delta T\right]^*\left(I + T^+\delta T\right)^{-1}T^+ \\ &= \left[I + \left(I + T^+\delta T\right)^{-1}T^+\delta T\left(I - T^+T\right)\right]^* T^+T \\ &\quad \Big\{I + \left[\left(I + T^+\delta T\right)^{-1}T^+\delta T\left(I - T^+T\right)\right] \\ &\quad \left[\left(I + T^+\delta T\right)^{-1}T^+\delta T\left(I - T^+T\right)\right]^*\Big\}^{-1}\left(I + T^+\delta T\right)^{-1}T^+,\end{aligned} \tag{5}$$

with

$$\frac{\left\|\tilde{T}^+ - T^+\right\|}{\|T^+\|} \le \frac{\|T^+\delta T\|}{1 - \|T^+\delta T\|}\left[1 + \frac{1}{(1 - \|T^+\delta T\|)^2}\right] \tag{6}$$

Furthermore, if in addition $\|T^+\|\|\delta T\| < 1$, then

$$\frac{\left\|\tilde{T}^+ - T^+\right\|}{\|T^+\|} \le \frac{\|\delta TT^+\|}{1 - \|T^+\|\ \|\delta T\|} + \|T^+\delta T\| \tag{7}$$

PROOF $R(\delta T) \subseteq R(T)$ implies that $\tilde{T} = T(I + T^+\delta T)$ and $\|T^+\delta T\| < 1$ implies that $I + T^+\delta T$ is bijective, and so $R(\tilde{T}) = R(T)$ and $\tilde{T}\tilde{T}^+ = TT^+$

By using the fact that

$$(I + UV^*)^{-1} = I - U(I + V^*U)^{-1}V^*$$

if either side is well defined, we can prove that the two representations of $\tilde{T}^+$ in (5) are the same

Let X be the right side of (5) The proof amounts to showing that the four Moore-Penrose equations (1) are satisfied First it is easy to verify that

$$(T+\delta T)X(T+\delta T) = TT^+(T+\delta T) = T+\delta T$$

Now, using the fact that $(I+T^+\delta T)^{-1}T^+\delta T = T^+\delta T(I+T^+\delta T)^{-1}$, we have

$$\begin{aligned}
X(T+\delta T) &= \left[I+\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*T^+T \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right\}^{-1} \\
&\quad \left(I+T^+\delta T\right)^{-1}T^+(T+\delta T) \\
&= \left\{T^+T\left[I+\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\right\}^* \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right\}^{-1} \\
&\quad \left(I+T^+\delta T\right)^{-1}T^+(T+\delta T) \\
&= \left[T^+T+T^+\delta T\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^* \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right\}^{-1} \\
&\quad \left(I+T^+\delta T\right)^{-1}T^+(T+\delta T) \\
&= \left[T^+T+\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^* \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right\}^{-1} \\
&\quad \left(I+T^+\delta T\right)^{-1}T^+(T+\delta T) \\
&= \left[\left(I+T^+\delta T\right)^{-1}T^+(T+\delta T)\right]^*\left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\right. \\
&\quad \left.\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right\}^{-1}\left[\left(I+T^+\delta T\right)^{-1}T^+(T+\delta T)\right] \\
&= [X(T+\delta T)]^*
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(T+\delta T)X &= T\left(I+T^+\delta T\right)\left(I+T^+\delta T\right)^{-1}T^+ + T\left(I+T^+\delta T\right)\left(I+T^+\delta T\right)^{-1} \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\right.\right. \\
&\quad \left.\left.\left(I-T^+T\right)\right]\right\}^{-1}\left(I-T^+T\right)\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\right]^*\left(I+T^+\delta T\right)^{-1}T^+ \\
&= TT^+ + T\left(I-T^+T\right)\left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right. \\
&\quad \left.\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]\right\}^{-1} \\
&\quad \left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\right]^*\left(I+T^+\delta T\right)^{-1}T^+ \\
&= TT^+ = [(T+\delta T)X]^*
\end{aligned}$$

Finally, we have

$$\begin{aligned}
X(T+\delta T)X &= \left(I+T^+\delta T\right)^{-1}T^+TT^+ + \left(I+T^+\delta T\right)^{-1} \\
&\quad \left\{I+\left[\left(I+T^+\delta T\right)^{-1}T^+\delta T\left(I-T^+T\right)\right]^*\right.
\end{aligned}$$

$$\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\left(I-T^{+}T\right)\right]\Big\}^{-1}$$
$$\left(I-T^{+}T\right)\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\right]^{*}\left(I+T^{+}\delta T\right)^{-1}T^{+}TT^{+}$$
$$=\left(I+T^{+}\delta T\right)^{-1}T^{+}+\left(I+T^{+}\delta T\right)^{-1}\Big\{I+\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\left(I-T^{+}T\right)\right]^{*}$$
$$\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\left(I-T^{+}T\right)\right]\Big\}^{-1}$$
$$\left(I-T^{+}T\right)\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\right]^{*}\left(I+T^{+}\delta T\right)^{-1}T^{+}=X$$

Hence, the four equations in (1) are satisfied From

$$\left(I+T^{+}\delta T\right)^{-1}T^{+}=\sum_{i=0}^{\infty}\left(-T^{+}\delta T\right)^{i}T^{+}=\left[I+\sum_{i=1}^{\infty}\left(-T^{+}\delta T\right)^{i}\right]T^{+}$$
$$=T^{+}-T^{+}\delta T\sum_{i=0}^{\infty}\left(-T^{+}\delta T\right)^{i}T^{+}$$
$$=T^{+}-T^{+}\delta T\left(I+T^{+}\delta T\right)^{-1}T^{+},$$

$$\tilde{T}^{+}-T^{+}=\left(-T^{+}\delta T\right)\left(I+T^{+}\delta T\right)^{-1}T^{+}+\left(I+T^{+}\delta T\right)^{-1}\Big\{I+\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\right.$$
$$\left.\left(I-T^{+}T\right)\right]^{*}\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\left(I-T^{+}T\right)\right]\Big\}^{-1}\left(I-T^{+}T\right)$$
$$\left[\left(I+T^{+}\delta T\right)^{-1}T^{+}\delta T\right]^{*}\left(I+T^{+}\delta T\right)^{-1}T^{+}$$

Since $\|(I+F^{*}F)^{-1}\|\leq 1$ for any bounded linear operator F, we arrive at

$$\frac{\left\|\tilde{T}^{+}-T^{+}\right\|}{\|T^{+}\|}\leq\frac{\|T^{+}\delta T\|}{1-\|T^{+}\delta T\|}+\frac{1}{1-\|T^{+}\delta T\|}\frac{\|T^{+}\delta T\|}{1-\|T^{+}\delta T\|}\frac{1}{1-\|T^{+}\delta T\|}$$
$$=\frac{\|T^{+}\delta T\|}{1-\|T^{+}\delta T\|}\left[1+\frac{1}{\left(1-\|T^{+}\delta T\|\right)^{2}}\right]$$

This proves (6) The last conclusion is from Remark 4 1 and (12) of [3] ∎

COROLLARY 2 *(See [3]) Let $T\in L(H_1,H_2)$ be injective with closed range, and $\tilde{T}=T+\delta T\in L(H_1,H_2)$ be such that $R(\delta T)\subseteq R(T)$ and $\|T^{+}\delta T\|<1$ Then $\tilde{T}$ is injective with closed range Moreover, $R(\tilde{T})=R(T)$ and $\tilde{T}^{+}=(I+T^{+}\delta T)^{-1}T^{+}=T^{+}(I+\delta TT^{+})^{-1}$ and*

$$\frac{\left\|\tilde{T}^{+}-T^{+}\right\|}{\|T^{+}\|}\leq\frac{\|T^{+}\delta T\|}{1-\|T^{+}\delta T\|}\tag{8}$$

THEOREM 3 *Let $T\in L(H_1,H_2)$ with closed range and $\tilde{T}=T+\delta T\in L(H_1,H_2)$ be such that $N(T)\subseteq N(\delta T)$ and $\|\delta TT^{+}\|<1$ Then $\tilde{T}^{+}$ is well defined, $\tilde{T}^{+}\tilde{T}=T^{+}T$, and*

$$\tilde{T}^{+}=T^{+}\left(I+\delta TT^{+}\right)^{-1}+T^{+}\left(I+\delta TT^{+}\right)^{-1}\left[\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]^{*}\left(I-TT^{+}\right)$$
$$\Big\{I+\left[\left(I-TT^{+}\right)\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]-TT^{+}\left[\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]$$
$$\left[\left(I-TT^{+}\right)\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]^{*}\Big\}^{-1}\left(I+\delta TT^{+}\right)^{-1}\tag{9}$$
$$=T^{+}\left(I+\delta TT^{+}\right)^{-1}\Big\{I+\left[\left(I-TT^{+}\right)\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]^{*}\left[\left(I-TT^{+}\right)\delta TT^{+}\right.$$
$$\left.\left(I+\delta TT^{+}\right)^{-1}\right]\Big\}^{-1}TT^{+}\left[I+\left(I-TT^{+}\right)\delta TT^{+}\left(I+\delta TT^{+}\right)^{-1}\right]^{*},$$

with

$$\frac{\left\|\tilde{T}^{+}-T^{+}\right\|}{\|T^{+}\|} \le \frac{\|\delta T T^{+}\|}{1-\|\delta T T^{+}\|}\left[1+\frac{1}{(1-\|\delta T T^{+}\|)^{2}}\right] \tag{10}$$

Moreover, if in addition, $\|T^{+}\|\|\delta T\| < 1$, *then*

$$\frac{\left\|\tilde{T}^{+}-T^{+}\right\|}{\|T^{+}\|} \le \frac{\|T^{+}\delta T\|}{1-\|T^{+}\|\,\|\delta T\|} + \|\delta T T^{+}\| \tag{11}$$

PROOF $N(T) \subseteq N(\delta T)$ implies that $\tilde{T} = (I + \delta T T^{+})T$ and $R((\delta T)^{*}) \subseteq R(T^{*})$ This theorem follows from $(T^{*})^{+} = (T^{+})^{*}$ and Theorem 2 applied to T^{*} ∎

COROLLARY 3 *(See [3]) Let* $T \in L(H_1, H_2)$ *be subjective, and* $\tilde{T} = T + \delta T \in L(H_1, H_2)$ *be such that* $N(T) \subseteq N(\delta T)$ *and* $\|\delta T T^{+}\| < 1$ *Then* $\tilde{T}$ *is subjective with closed range Moreover,* $N(\tilde{T}) = N(T)$ *and* $\tilde{T}^{+} = (I + T^{+}\delta T)^{-1}T^{+} = T^{+}(I + \delta T T^{+})^{-1}$ *and*

$$\frac{\left\|\tilde{T}^{+}-T^{+}\right\|}{\|T^{+}\|} \le \frac{\|\delta T T^{+}\|}{1-\|\delta T T^{+}\|} \tag{12}$$

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